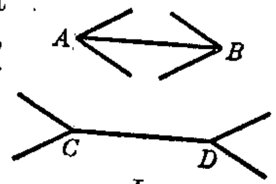
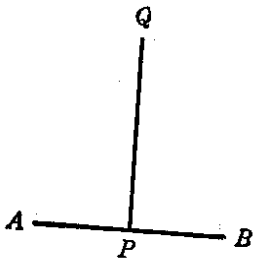
FUNDAMENTALS AND AXIOMS OF GEOMETRY

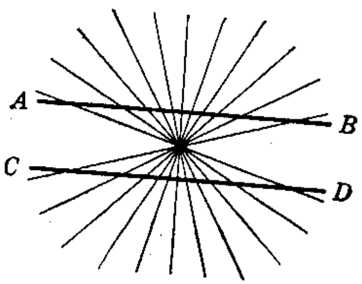
42. Experimental and demonstrative geometry. If we make an accurate construction of a figure and then make careful observations, it is possible to discover many geometric facts. However, it is physically impossible to have all of the constructions absolutely accurate and the eye can readily be misled so that we are likely to make incorrect judgments concerning figures. Thus, in Fig. I, which is the longer, AB or CD? In Fig. II which is the longer, AB or PQ? In Fig. III are the lines AB and CD straight?



I



II



III

For these and other reasons we shall not depend upon experimental methods in our study of geometry. We shall start with a few fundamental assumptions and from these deduce other true statements. No statements except the fundamental ones will be accepted as true until they have been proved. This type of geometry

is called *demonstrative geometry*. The statement that is to be proved is called a *theorem*.

43. Fundamental assumptions. There are two distinct kinds of fundamental assumptions: (1) *axioms;* (2) fundamental propositions which are sometimes called *postulates*. An axiom is an assumption which refers to quantities in general, while a fundamental proposition or postulate refers primarily to geometric figures.

Axioms:

1. *If equals are added to equals, the results are equal. Thus, if* $a = b$ *and* $x = y$*, then* $a + x = b + y$.
2. *If equals are subtracted from equals, the results are equal. Thus, if* $a = b$ *and* $x = y$*, then* $a - x = b - y$.
3. *If equals are multiplied by equals, the results are equal. Thus, if* $a = b$ *and* $x = y$*, then* $ax = by$.
4. *If equals are divided by equals, the quotients are equal, provided the divisors are not zero.*†

   Thus, if $a = b$ *and* $x = y \neq 0$*, then* $\frac{a}{x} = \frac{b}{y}$.
5. *The whole of a quantity is equal to the sum of its parts.*
6. *The whole of a quantity is greater than any of its parts.*
7. *Quantities equal to the same quantity are equal to each other — i.e., equals may be substituted for equals.*

† If division by zero were possible, then any two numbers would be equal. For example, $78 \times 0 = 0$, $456 \times 0 = 0$, and hence $78 \times 0 = 456 \times 0$. Dividing both members by zero we have $78 = 456$.

EXERCISES

Answer the following questions, and in each case state which axiom supports your answer.

1. Mary is just as old as Jane. How will their ages compare ten years hence? How did their ages compare two years ago?

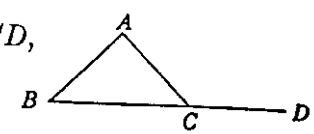
2. At the end of the first quarter, the Cornell and Dartmouth teams were tied. At the end of the game, each team had tripled its first quarter score. How did the game end?

3. Mr. Martin and Mr. Nardini each sold the same number of television sets in September. However, October was a slack month, and each sold only half as many sets as he had in September. Did Mr. Martin sell as many sets in October as Mr. Nardini?

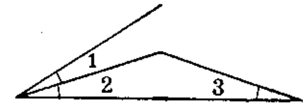
4. Several apple pies of the same size were each cut into six equal slices. Did a person who ate six pieces consume a whole pie? Before one of the pies could be cut, Jerry ate all of it except a small sliver that he gave his father. Could one say accurately that Jerry ate a whole pie?

5. Louise and Evelyn weighed themselves one day on a penny scale and found their weights were the same. The next day Louise and Joyce used the same scale. They found that Joyce also weighed the same as Louise. How did Joyce and Evelyn compare in weight?

6. If $AC = AB$, and $AC = CD$, then $AB = CD$.



7. If $\angle 1 = \angle 2$, and $\angle 2 = \angle 3$; then $\angle 1 = \angle 3$.



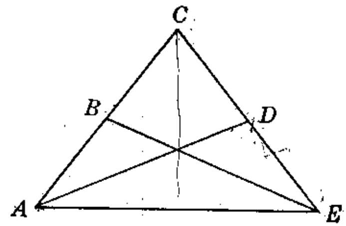
Ex. 7.

8. If $AB = ED$, and $BC = DC$; then $AC = EC$.

9. If $AC = EC$, and $AB = ED$; then $BC = DC$.

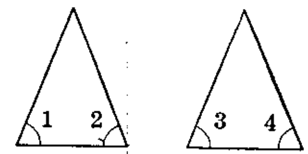
10. If $AB = \frac{1}{2}AC$, and $ED = \frac{1}{2}EC$, and $AC = EC$; then $AB = ED$.

11. If $AC = 2AB$, and $EC = 2ED$, and $AB = ED$; then $AC = EC$.



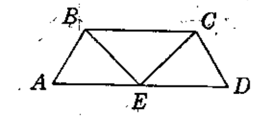
Ex. 8–11.

12. If $\angle 1 = \angle 3$, and $\angle 2 = \angle 4$, and $\angle 1 = \angle 2$; then $\angle 3 = \angle 4$.



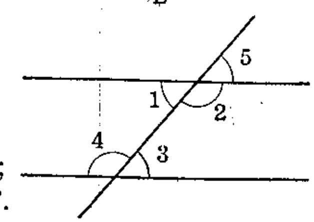
Ex. 12.

13. If $\angle ABC = \angle DCB$, and $\angle ABE = \angle DCE$; then $\angle EBC = \angle ECB$.



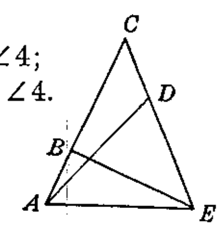
Ex. 13.

14. If $\angle 1 = \angle 5$, and $\angle 1 + \angle 2 = \angle 3 + \angle 4$; then $\angle 2 + \angle 5 = \angle 3 + \angle 4$.



Ex. 14.

15. If $AB = CD$, and $BC = ED$; then $AC = EC$.



Ex. 15.

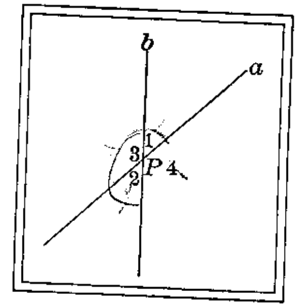
44. The fundamental propositions or postulates we shall use are:

1. *Through any two distinct points it is possible to draw one and only one straight line, and therefore two distinct straight lines can intersect in only one point.*
2. *A line segment can be produced to any desired length.*
3. *A figure can be moved from place to place without changing its size or form.*
4. *A segment has but one mid-point.*
5. *In a given plane one and only one circle can be drawn having a given center and a given radius.*
6. *A straight line intersects a circle in not more than two points.*
7. *All right angles are equal.*
8. *One and only one perpendicular can be drawn from a point to a line.*
9. *The perpendicular from a point to a line is the shortest distance from the point to the line.*
10. *One and only one perpendicular can be erected to a given line at a given point on the line.*
11. *Equal angles have equal complements and equal supplements — i.e., the complements of the same or equal angles are equal, likewise the supplements are equal.*
12. *If two adjacent angles have their exterior sides in a straight line, the angles are supplementary.*

45. Before proceeding to the main set of theorems we shall illustrate the methods of demonstrative geometry by proving an important theorem. In studying this theorem, the student should take particular pains to notice the form in which it is written.

(1)

46. *Theorem.* *If two straight lines intersect, then the vertical angles are equal.*



Given the straight lines a and b intersecting in the point P and forming the pairs of vertical angles 1 and 2, 3 and 4.

To prove that $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.

Analysis: If we can prove $\angle 1$ and $\angle 2$ supplements of the same angle, they will be equal.

Proof:

1. $\angle 1 + \angle 3 = 180°$.	1. F.P. 12. If two adjacent angles have their exterior sides in a straight line the angles are supplementary.
2. Likewise $\angle 2 + \angle 3 = 180°$	2. Same reason as for 1.
3. $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 3$.	3. Things equal to the same thing are equal to each other. (Axiom 7.)
4. Hence $\angle 1 = \angle 2$.	4. If equals are subtracted from equals the results are equal. (Axiom 2.)
5. In the same manner we can prove $\angle 3 = \angle 4$.	

Example. If one of the four angles formed by two intersecting lines is a right angle, what is each of the other three angles?

47. Arrangement of a demonstration. It will be noted that there were four steps involved in the demonstration of the theorem just given:

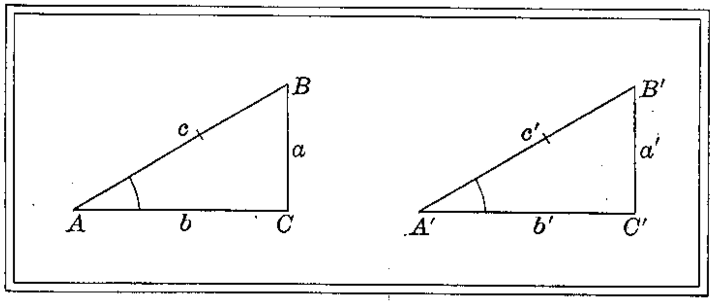
1. The statement of what is given, which is called the *hypothesis*.
2. The statement of what is to be proved, which is called the *conclusion*.
3. An *analysis* or *plan* of the method of procedure whereby we discover what to do in order to reach our conclusion. In making an analysis we say to ourselves, I can prove A if I can prove B; I can prove B if I can prove C; and so on until we arrive at something we can prove. If we now reverse our steps, we have the method of proof.
4. The *proof*, which consists of several steps each supported by some authority such as a definition, a fundamental proposition, or a previously proved theorem. These four steps are indicated by the words

Given, To prove, Analysis, Proof.

Every demonstration should be arranged in this way. We shall give another example, before beginning the more systematic development of the subject of geometry.

It should also be noted that in the proof the facts are given on the left and the corresponding reasons on the right. Students should always state their reasons in words.

48. *Theorem.* *If two right triangles have the hypotenuse and an acute angle of the first equal respectively to the hypotenuse and an acute angle of the second, then the triangles are congruent.*



Given the two right triangles ACB and $A'C'B'$, with $c=c'$, $\angle A=\angle A'$, and right $\measuredangle$ C and C'.

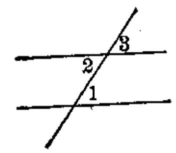
To prove that $\triangle ACB \cong \triangle A'C'B'$.

Analysis: Congruent ⚠ are those which can be made to coincide. Therefore we shall place one $\triangle$ upon the other and show that they coincide.

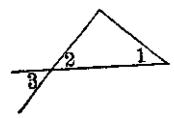
Proof: Place $\triangle ACB$ on $\triangle A'C'B'$ so that $\angle A$ coincides with $\angle A'$. Then b will fall along b' and c will fall along c'.

1. Point B will fall on point B'.	1. For $c=c'$ by hyp.
2. Both a and a' are $\perp$ to the base b.	2. Hyp.
3. $\therefore$ a and a' coincide.	3. Only one perpendicular can be drawn from a point to a line.
4. Hence $\triangle ABC \cong \triangle A'B'C'$.	4. Congruent ⚠ are those which can be made to coincide.

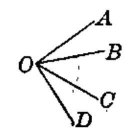
EXERCISES



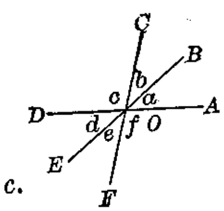
1. Given
$\angle 1 = \angle 2.$
Prove that
$\angle 1 = \angle 3.$



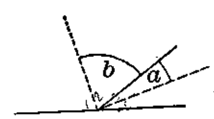
2. Given $\angle 1 = \angle 2.$
Prove that
$\angle 1 = \angle 3.$



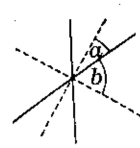
3. Given
$\angle AOC = \angle BOD.$
Prove that
$\angle AOB = \angle COD.$



4. Prove that
(a) $\angle DOF = \angle a + \angle b.$
(b) $\angle EOC - \angle a = \angle DOC.$
(c) $\angle DOB + \angle BOF - \angle FOD = 2\angle c.$



5. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.



6. Prove that the bisectors of the pairs of vertical angles formed by the intersection of two lines are perpendicular.
(Prove $\angle a + \angle b = 90°$.)

7. Prove that the sum of the angles about a point in a plane is equal to 360°.

THE "IF-THEN" RELATIONSHIP

In the two theorems you have studied, you may have noticed that each contains an "if" clause and a "then" clause. The "if" clause contains the given conditions or hypothesis; the "then" clause contains the conclusion to be proved true for all cases where the given conditions exist.

Any theorem may or may not be stated in the "if-then" form. When it is not, rephrasing the theorem into an "if-then" sentence often helps in organizing and planning a demonstrative proof. Suppose the first theorem, on page 63, had been stated thus: "The vertical angles formed by the intersection of two straight lines are equal." To distinguish between what facts were given and what was to be proved might then have been more difficult.

In Exercise 5 on the opposite page, you were asked to prove that "the bisectors of two supplementary adjacent angles are perpendicular to each other." Changing this into the "if-then" form, we get "if two supplementary adjacent angles are bisected, then the two bisectors are perpendicular to each other." Then it becomes easy to determine what is given and what is to be proved:

Given two supplementary adjacent angles bisected so that $2\angle a + 2\angle b = 180°$.

To prove that the bisectors (the dotted lines in the figures) are perpendicular to each other; i.e., that $\angle a + \angle b = 90°$.

Whenever, in the future, you have trouble identifying the hypothesis and conclusion, try transforming the statement into the "if-then" form.

REASONING: INDUCTIVE AND DEDUCTIVE

We use both inductive and deductive reasoning constantly in our daily life, although we shall use the latter almost entirely in studying geometry. Since it is important to think clearly no matter which type of reasoning we use, we should understand the two types and know the dangers to avoid in each if we are to draw intelligent conclusions.

Inductive reasoning. In Exercise 10 on page 20, you measured the size of each angle in three equilateral triangles, and probably concluded that any angle in any equilateral triangle always contains 60°. The type of reasoning you used is called inductive reasoning. It is very important to note that you merely established the probability that the angles of an equilateral triangle are 60° angles. You did not prove it absolutely because to do so inductively would require that you measure every angle of every possible equilateral triangle, something you could not possibly do if you did nothing else the rest of your life. (Yet, when you reach page 157 of this text, you will be able to prove it conclusively by deductive reasoning.)

Suppose a chemist took several samples of water from different places in a lake, tested these samples, found the water in each sample to be pure, and then announced, "The water in the lake is pure." Has the chemist actually proved that all the water in the lake is pure, or has he merely established the probability that it is?

Conclusions we derive from experience, or general laws based on a number of experiments, are examples of inductive reasoning. Scientists, to date, have found that the temperature of every gas they have compressed has risen. Hence, they say, "All gases become hotter when compressed." However, if one gas should be found which did not become hotter when compressed, the above statement would be proved false.

Because it is easy to be inaccurate in our observations and because it is impossible to achieve complete observations of every pertinent fact, the truth of conclusions reached inductively is bound to be somewhat uncertain. The probable truth of such a conclusion depends on the number and accuracy of the observations. Consequently, before reaching a conclusion by inductive reasoning, we should be as sure as possible that enough facts have been observed accurately and that no fact exists which does not fit our conclusion.

EXERCISES

In each of these exercises, explain why the conclusion drawn is a questionable one.

1. Each of the five bull dogs John has seen have been vicious. He therefore says, "I would not own a bull dog, because they are vicious."

2. Mrs. Brown found that scarlet poppies, scarlet verbenas, scarlet hawthorn, and scarlet honeysuckle have no odor. She therefore decided that all scarlet flowers are odorless.

3. Mr. Burns measured his backyard by placing one foot in front of another, and announced that his yard was exactly 42 feet by 24 feet.

4. A candidate for political office implied that because his opponent had once voted against farmers' interests, he would always do so.

5. Mrs. Payne carefully avoided shopping at a certain dress shop because she had once bought a dress there which had not worn well.

Deductive reasoning. If we start by accepting certain beliefs, called *premises*, as being true, and then use these premises to arrive at a new and necessary conclusion, we say we have reasoned by deduction. In proving our first geometric theorems, we reasoned deductively. The reasons given at the right of the proof were our premises, and our successive conclusions were given at the left. Note that the premises consisted of the definitions and fundamental assumptions previously accepted as true.

The truth of conclusions reached deductively depends on the truth of the premises and on the correctness of the reasoning. Examine these two examples of deductive reasoning. The first two statements in each case are the premises, and the last is the conclusion.

1. All cows have four legs.
   Rover has four legs.
   Hence, Rover is a cow.

2. Only cows have four legs.
   Rover has four legs.
   Hence, Rover is a cow.

In the first example, the premises are true, but the reasoning is incorrect, since the conclusion does not inescapably follow from the premises. The first premise does not prevent the possibility of other animals having four legs also. In the second example, the reasoning is correct, but the first premise is not acceptable as being true.

In the following example, the premises are true and the reasoning is correct, so the conclusion *must* be true.

All cows have four legs.
A Holstein is a cow.
Hence, a Holstein has four legs.

It is important to note that a conclusion may be true even though both premises are untrue and the reasoning is incorrect, as in the following deduction.

Only men write popular novels.
George Eliot was a man.
Hence, George Eliot wrote popular novels.

The first premise is obviously false. The second premise looks true but is not, because George Eliot was the pen name of Mary Ann Evans, English novelist of the 1800's and author of *Silas Marner*.

EXERCISES

In each of these exercises, decide whether the argument contains untrue premises, incorrect reasoning, or both.

1. All Texans are tall and Gary Cooper is tall, so Gary Cooper must be a Texan.

2. Since all people who wear glasses are near-sighted, and that girl wears glasses, she must be near-sighted.

3. All dictionaries contain definitions of words. This book gives word-definitions, so it must be a dictionary.

4. Eating too much makes one overweight. John is overweight, so he must have been eating too much.

5. The football coach said that any player caught smoking would be dismissed from the squad. John was dismissed from the squad, so he must have been caught smoking.

Unstated premises and conclusions. Many people fail to think clearly because they neither understand nor analyze the premises and reasoning they use every day. If you are to improve your reasoning power and make important decisions more intelligently, you must form the habit of testing the truth of your assumptions and the correctness of your reasoning before drawing conclusions.

It is especially important to be on guard against arguments in which assumptions and even conclusions are purposely omitted, in hopes that the reader or listener will not recognize the untruth of the premises or the incorrectness of the reasoning. For example, a famous movie star is advertised as using a certain brand of soap. The advertiser hopes you will accept without examination the unstated and questionable assumption that using that

particular soap was an important factor in making her beautiful. He also hopes that you will jump to the implied but illogical conclusion that because it helped make her beautiful, it will do the same for you.

EXERCISES

In these exercises, identify and evaluate the unstated assumptions and conclusions that have been made or implied.

1. Mrs. Bond found her son's temperature was 103°. She telephoned her husband and said, "Bob has pneumonia."

2. James believes that because he took the business course he is qualified for any business position.

3. Ralph, on a trip to the city, took the shortest route because he was in a hurry to get there.

4. Mr. Richards bought the newer model of two used cars even though it had been driven much farther and harder than the older model.

5. In an advertisement, a well-known baseball player states that he always eats Munchies for breakfast.

Look Here

HOW TO STUDY GEOMETRY

1. ***Proceed as though no proof were given.*** *Place a sheet of paper over the figure and the proof so you can see only the statement of the theorem.*

2. ***Divide the theorem into what is given and what is to be proved.***

3. ***Draw the figure as general as possible.*** *For example, if a triangle is given, do not draw a special triangle such as one with equal sides.*

4. ***Write down the hypothesis and conclusion*** — *that is, what is given as applied to your figure and what is to be proved.*

5. ***Classify the theorem under some general topic.*** *For example, ask yourself, Does this come under the topic of proving triangles equal, lines equal, angles equal, lines parallel, and so on? Keep clearly in mind what it is you wish to prove.*

6. ***Suppose you wish to prove two angles equal by congruent triangles.*** *Analyze the proposition as follows:*
   I can prove the angles equal if I can prove the triangles congruent.
   The triangles will be congruent if I can show that SAS = SAS, etc.
   One pair of equal sides is AB and A′B′ because . . .
   . . . .

Another pair of equal sides is BC and B′C′ because . . . .

One pair of equal angles is B and B′ because . . . .

The triangles are congruent because. . . . .
The angles are equal because. . . . .

7. ***Then write out the steps in order.***

8. ***Finally read the proof in the book and compare your proof with it.*** *If you adopt this method of study, you will save time in the end, you will enjoy discovering the proof yourself, and you will really learn how to think clearly and logically.*

The Word ANY

In geometry, the word "any" is used in the sense of "every." For example, if we are asked to prove a theorem concerning *any* triangle, we must not select a special type of triangle such as an isosceles or equilateral triangle, but must choose the most general type, i.e., a triangle with no two sides or angles equal, called a *scalene* triangle. Again, suppose we are asked to connect any point in the base of an equilateral triangle to the opposite vertex. Here we must not choose the mid-point nor an end point of the base, for these are particular points. Any other point in the base will represent a general situation and must be chosen. A *special case never represents all cases.*