Sets and Real Numbers

Appendix E - Set Notation and the Real Numbers

Points are Real Numbers

- We will re-build the foundation of geometry on the foundation of sets and real numbers.
- Almost all mathematics is currently organized on the basis of axioms of Set Theory
- Real Numbers are constructed assuming the axioms of Set Theory
- We add axioms for the Real Numbers that give us all the standard laws of algebra that we commonly use
- By building the axioms on Sets and Real Numbers we greatly simplify rigorous proofs of Euclid's propositions
- See the Hilbert axioms in Appendix B for a rigorous foundation for Euclid's geometry that doesn't assume points as Real Numbers.

Sets

- The words *set* and *element* are undefined terms in Set Theory
- We intuitively think of sets as collections of elements (the axioms of set theory make this intuition rigorous)
- The element-of operation looks like a round-back *E*
- Sets are specified by two general ways illustrated in Apx E
    - Listing the elements
    - Giving a rule
- The empty set, or null set, is a special set with no elements
- Comparing sets
    - Subset (the contained-in operation)
    - Proper subset
    - Set equality

Set Operations and Sets of Numbers

- Set Operations
  - Union
  - Intersection
  - Difference

- Standard sets of numbers
  - **N** - Natural Numbers (sometimes with 0, sometimes not)
  - **Z** - Integers
  - **Q** - Rational Numbers
  - **R** - Real Numbers
  - **C** - Complex Numbers
  - R\Q - Irrational Numbers (Reals minus Rationals)

Real Number Representation

- We commonly think of the Real Numbers as the collection of all possible decimal representations, including those that do not terminate (Eg: pi = 3.14159265358979...)
- Decimal representation is a base 10 positional notation that we have come to fully understand only very recently in the history of mathematics
- Decimal representation may have been the single most important technological invention in all of mathematics
- Rational Numbers all have either finite decimal representations or infinite but repeating decimal expansions
  - 6.78
  - 18.345345345...
- Irrationals all have non-repeating infinite decimal representations

Real Number Axioms

- We assume without discussion all the common laws of the algebra of Real Numbers (identity, distributivity, etc)
- The Real Numbers are composed of the Rationals plus all the Irrationals (like sqrt(2)) that fill in the gaps in the Real Plane
- Put another way, we have both Rational and Irrational points
- The **Trichotomy Postulate** (see Venema Apx E)
  - This gives us the ordering and betweenness properties that we need in geometry
- The **Density Postulate** (see Venema Apx E)
  - This tells us that there are always points between any two points on a line
  - It also tells us that the irrational and rational points are arbitrarily mixed up with each other in a line

The Comparison Theorem

- This isn't an axiom, but it is one of the more important and useful properties that we get from the Trichotomy and Density Postulates
- Real Numbers are not easily compared because many of them have infinite decimal expansions, so they cannot be finitely compared digit by digit
- All properties of the Reals that have to do with infinite representations are founded on the concept of a limit in the calculus
- Intuitively and very roughly the concept of a limit lets us conclude (with our required mathematical rigor) that we "have" a number with an infinite expansion if we can get as close as we we want to it - with any specified degree of precision.

Using the Comparison Theorem

- When we work with Real Numbers we can at best work with Rational Number approximations to Real Numbers
- If we want to compare Real Numbers, then we need to express the comparison in terms of all possible rational number approximations that might arise (to any degree of precision)
- Intuitively, we do this in the Comparison Theorem by saying that $x = y$ if we can demonstrate that no rational approximation (of any real number to any degree of precision) can possibly get between x and y
- The theorem is stated in a way that makes it easier to use when doing proofs - we know what we need to do to show that a real number x equals a real number y

The Archimedean Property of Reals

- This tells us the relationship between the ordering of the integers, which are discrete, and the ordering of the Reals that represent the Continuum.
- Imagine that you have some large Real M representing a length that you want to measure with some tiny ruler of length epsilon
  - Then this axiom tells us we can always cover the length M with a whole number of measurements using our epsilon ruler.
- Alternatively, the axiom tells us that for any Real number K, there is a natural number n larger than K
- Again, the axiom is stated in a way that us useful for the proofs in which it is needed
- This property follows from the Least Upper Bound Postulate

The Least Upper Bound Postulate

- This can be intuitively viewed as supplying the continuity property for the Reals and thus for the Real line and the Real plane R^2 of points.
- It says that the Reals are not only dense but have no gaps
- It solves one of Euclid's problems by guaranteeing the existence of points at intersections
- We'll need this when we need continuity,
- The property goes beyond the Archimedean Property which tells us we can always get there - now we know we won't fall into a gap on the way!

Functions

- Functions along with Sets provide the essential tools of mathematics
- Functions are rules that assign to each element x of a **Domain** set A, a corresponding and unique element f(x) of a **Codomain** set B. A codomain is sometimes called the Range
- For a function f from A to B we write f: A -> B
- We say a function f *maps* every element in A to a unique element in B
- For x in A, we say the element f(x) in B is the *image* of x
- Every element in A has some image in B
- Not every element in B is an image of an element in A
- The specification of a function *requires that you identify the Domain and Codomain*, even though they are not often explicitly stated in math books (they should be).

One to One Correspondence

- We say a function f:A->B is a *one to one correspondence* or a *bijection* between A and B if f matches up each element in A with a distinct element in B and with no left over elements in B
- Note that A and B must have exactly the same number of elements in order to have a one to one correspondence between them
- Note also that if we have a one to one correspondence between A and B then we also have a one to one correspondence between B and A via a function called the *inverse* of f, denoted f^{-1}.

One to One Mapping

- It is possible to match up elements from a set A with another set B even if A and B are not the same size.
- If B has more elements than A, then a function f:A->B can match up each element of A with a unique element of B but leaving some elements of B with no match.
- When we match up A and B in this way with leftover B elements, f is a *one to one* function
- Note that the only difference between a one to one correspondence and a one to one function f:A->B is the leftover elements of B
- We can always construct a one to one correspondence from a one to one function f:A->B by changing the codomain set B of the function f to leave out all the leftover elements

Onto Functions

- Think about a mapping f:A->B in which every element of B is an image of some element in A.
- Such a function f:A->B is called *onto* if the mapping leaves out no element of B
- We don't care about the one to one property of f, we just want to make sure there are no leftover elements in B
- Note that in this case the codomain set B must be the same size or smaller than the set A

- **Remember** - a function f:A->B cannot map any element of the domain A to two different elements in B - that would be indeterminate and the definition of a function disallows this.