Euclid's Elements

Chapter 1 - Euclid's Elements

Geometry Before Euclid

- Before the the Greeks
  - Geometry - measuring the earth
  - Advanced body of practical geometric knowledge
  - Example - Pythagorean law was well known in practice
- The Presocratic Greeks 600 - 400 BCE
  - Thales - deductive reasoning, early science principles
  - Pythagoreans - all is number
  - Parmenides, Zeno - all is one
  - Heraclites - all is change
- Plato and Socrates 400 BCE
  - Senses do not reveal the truth
  - Truth is mathematical
  - Truth is recalled from your immortal soul

Geometry after Euclid

- Euclid 300 BCE
  - Abstraction
  - Logical Deduction
  - Reasoning from First Principles
  - Rational Numbers (whole numbers and their ratios)
- Geometric objects like points, lines, triangles
  - are pure and ideal forms with no material essence (abstractions)
  - they float in the plane with no fixed location
  - they are not identified with coordinate locations
  - they have properties like magnitude, but they are not associated with numbers
  - Example: lines were not lengths because some lines did not have lengths like the diagonal of a 1x1 square

Coordinate Geometry after Descartes

- Descartes 1600 CE
  - Cartesian coordinate system for the plane
  - Geometric forms become associated with numbers (eg, line and length)
  - Geometric forms are seen to exist with (x,y) locations in the plane
  - Decimal number system takes hold in Europe
  - Algebra begins

Reasoning from First Principles

- Logical Structure of Euclid's Elements
  - Definitions
  - Postulates - self-evident truths about constructions in space
  - Common Notions - self-evident truths about equality and comparison
  - Propositions - truths derived deductively by strict logical reasoning from the definitions, postulates, common notions, and prior propositions
- Euclid is considered the master compiler of the known results into a rigorous system of knowledge based on the absolute certainty of deductive reasoning - proof.
- We call this an axiomatic system of knowledge

Euclid's Fifth Postulate

- Much longer than the others
- Fairly complicated
- Not intuitively obvious
- Lines produced indefinitely (what does that mean?)
- Tries to avoid "infinity" of a line as we think of it
- Is about parallel lines, but formulated negatively
- For 2000 years mathematicians tried to prove the fifth postulate or replace it with a simpler version
- "Attempts to prove Euclid's Fifth Postulate eventually resulted in the realization that, in some kind of stroke of genius, Euclid has the great insight to pinpoint one of the deepest properties that a geometry may have".

Non-Euclidean Geometry

- "It was discovered in the 19th century that there are alternative geometries in which Euclid's Fifth Postulate fails to hold" - *non-Euclidean geometries*.
- "These discoveries opened up whole new fields of mathematical study"
- They produced a revolution of how mathematics relates to the real world
- They forced a new understanding of the nature of truth and the material world
- The Foundations of Geometry can only be properly understood in the light of the story the discovery of Non-Euclidean Geometries
- This course and book is about that story

A Look at Euclid's Elements

- Definitions 1,2,4,10,11,12
- Postulates 1 to 3 for constructions
- Postulate 4 on the equality of right angles (a technical necessity)
- Postulate 5 about parallel lines
- Common Notions 1-3 about equality
- Common Notion 4 about congruence as equality
- Common Notion 5 about relative magnitude

The common notions cover what we now think of as equality and order in the algebra number systems.

The Greeks did not have real numbers. They only had ratios of whole numbers (rational numbers).

Proposition 1

On a given finite straight line to construct an equilateral triangle.

- How do we know the point C intersecting circles exists?
- Illustrates possible holes in Euclid's continuum
- Euclid's drawings appeal to intuition and allude to continuity
- Deductive basis was not there as an explicit axiom - a continuity axiom is needed
- Euclid's rational numbers had holes in the continuum
- See figure 3.7 p45 and example 3.2.6 p44

Proposition 4 - SAS congruence

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

- Congruence is the equality of geometric objects
- Euclid uses the method of superposition is used for congruence: "... triangle ABC be applied to triangle DFE"
- Euclid really needs an axiom for the method of superposition
- The ability to move objects without changing shape cannot be taken for granted

Proposition 16 - Exterior Angle Theorem

In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

- How do we know the constructed point F is in the interior of the angle?
- A demonstration of this fact is necessary - axioms on betweenness and interiority are needed
- Example: Hilbert's axioms of order in Apx B satisfy betweenness and interiority
- How do we know the midpoint E exists and is unique?
- Euclid needs an axiom for the existence and uniqueness of midpoints
- The real number axioms of Apx E provide betweenness and interiority in a natural analytical (numerical) way

A Critique of Euclid's Element

- In light of contemporary standards of mathematical rigor
- Definitions are not self-evident truths, but starting places - just what is "that which has no part" (point)?
- All axiomatic systems must have some undefined terms
- Comparisons of magnitudes needed (an acute angle is less than a right angle - what does that mean?)
- Existence must be demonstrated, not just drawn. How do we know the point C for Proposition 1 exists?
- Theory of congruence and equality needs to be explicit - Euclid's *method of superpositon*.
- Uniqueness as well as existence must be demonstrated.
- Between-ness and interiority must have explicit axioms.
- Proofs should not rely on diagrams for existence and relative location.

Points and Numbers

- For Euclid, points were not numbers
- A line segment is an infinite number of indivisible points. Just what does that mean?
- We think of points as numbers since Descartes
- Euclid never considers the existence of an infinite extension of an object like a line (the entire infinite line at once). Lines in Euclid can always be extended to be as long as is needed.
- Euclid had no protractor or distance measure. We base our geometry now all on measurement and real numbers.
- Euclid identifies construction with existence - we no longer accept that
- We will re-build geometry on the real number system with points as numbers