Interpretations and Models

Chapter 2.1-2.4 - Axiomatic Systems and Incidence Geometry

Axiomatic Systems in Mathematics

- The gold standard for rigor in an area of mathematics
- Not fully achieved in most areas of mathematics
- Russel and Whitehead tried in early 20th century to axiomatize all of mathematics
- Goedel proved a complete axiomatization is not possible
- Geometry is the area in which the axiomatic approach was most successful
- A study of geometry using the axiomatic approach is called synthetic geometry
- Axiomatic systems are now really a branch of mathematical logic rather than mathematics in general
- Axiomatic systems are a natural component of computer science and computational representation of knowledge

Components of Axiomatic Systems

- Undefined terms
- Definitions of defined terms
- Axioms
  - We use axiom and postulate interchangeably
  - Axioms give meaning to undefined terms
  - Axioms are fundamental assumptions without proof
  - Everything else is logically deduced from the axioms
  - The only properties of definitions that can be used are those spelled out in the axioms
- Theorems (lemmas and corollaries too)
  - Theorem and proposition mean the same thing
  - Theorems must be deduced by valid rules of reasoning
  - Only axioms and previously proved theorems can be used

Interpretations

- An *interpretation* gives specific meaning to undefined terms
- Undefined terms may be interpreted in any way that is consistent with the axioms
- Euclidean geometry was "truth" about the physical space, so there were no alternative interpretations
- Non-Euclidean geometry opened up the possibility of alternative interpretations for an axiomatic system
- Non-Euclidean geometry also suggested that physical space may not be (as flat) as we think it is.

Models

- An interpretation is called a *model* for the axiomatic system if the axioms are true statements in that interpretation
- All theorems are logically deduced from the axioms, so all theorems are automatically true (correct) in that interpretation
- A model is a system of entities that behave in all ways according to the rules or laws of the axiomatic system

Consistency of an Axiomatic System

- The axioms in an axiomatic system are said to be *consistent* if no logical contradiction can be derived from them
- A contradiction is a statement that is both true and false at the same time (eg: A and (not A))
- If you can show there exists a model for an axiomatic system then it must be consistent

Independent Statements

- A statement in an axiomatic system is *independent* of the axioms if it is impossible to either prove or disprove the statement as a logical consequence of the axioms
- You can show a statement is independent of the axioms if you can give two distinct models - one in which the statement is true and the other in which the statement is false
- Euclid's fifth postulate was eventually determined to be independent of the other postulates
- We can build a Euclidean model in which the fifth postulate is true
- And we can build a Non-Euclidean model in which the fifth postulate is false (actually two distinct models)

Axioms for Incidence Geometry

Undefined terms

- point
- line
- lie on (incident with)

Axioms

1. For every pair of distinct points P and Q there exists exactly one line l such that both P and Q lie on l
2. For every line l there exist at least two points P and Q such that both P and Q lie on l
3. There exist three points that do not all lie on any one line, that is, there exists three noncollinear points

Three points A, B, C are *collinear* if there exists one line l such that all three points A, B, and C all lie on l.

Theories and Models for Incidence Geometries

The axiomatic system with the three undefined terms and the three axioms above is called an Incidence Geometry

We also usually call a model for the axiomatic system an Incidence Geometry

Note that the axiomatic system and the model are distinct mathematical systems. The axiomatic system is just a set of rules. The models are "real" things that obey those rules.

An axiomatic system fully characterizes some "real" system - even if we don't know what that "real" system is. Conversely, we have a "real" universe for which physicists are seeking an axiomatization - the theory of everything!

The three-point plane geometry

- Interpret point to mean one of three symbols A, B, C
- interpret line to mean a set of two points
- interpret lie on to mean "is an element of"

This is Example 2.2.2. There are three lines: {A,B}, {A,C}, {B,C}
The three-point plane is a finite geometry

Show that this is a model for incidence geometry - demonstrate that all three axioms are true in this interpretation.

Note that the drawing for this and other finite interpretations are misleading because the drawn "lines" are really just sets of the endpoints. The "line" part of the drawing is not part of the interpretation!

Other Finite Interpretations of Incidence Axioms

- Four-point geometry (p21)
- Fano's geometry (p22)
- Three point line (p21) - not a model

Show that the first two are models and show why the third is not a model

To show an interpretation is not a model you need only find one counterexample to one of the axioms

The three point line does not satisfy the Incidence Axiom 3

To show that an interpretation is a model, you essentially have to prove that there are no counterexamples

The Cartesian plane geometry

- This one you know well. It is based on the real number plane which we denote R^2
- Interpret point to be a pair of real numbers (x,y)
- Interpret line to be the collection of points whose coordinates satisfy a linear equation of the form $0 = ax + by + c$ where a, b, and c are real numbers and not both a and b are 0
- Note that $ax + by + c = 0$ is $y = -(a/b)x - (c/b)$
- We could just say a line is the set of all pairs of points (x,y) in R^2 such that $y=mx + b$
- The Cartesian plane is obviously not a finite geometry

The sphere

- Interpret point to mean a point on the surface of a 2-sphere in three-dimensional space.
- Interpret line to be a great circle - a circle on the surface whose radius equals the radius of the sphere
- We denote the sphere by S^2
- Note that we are only talking about the surface of the sphere when we talk about S^2
- The sphere is not a model for the incidence axioms - why?
- There are no parallel lines in the sphere - all lines intersect
- The sphere is a non-finite geometry, but it is a bounded geometry!

The Klein disk

This one is fun. We'll come back to it later in the quarter

The Klein disk is a non-finite geometry, but it is a bounded geometry!

The parallel postulates in incidence geometry

Two lines l and m are *parallel* if there is no point P such that P lies on both l and m, denoted m || l (note: a line is not parallel to itself)

Euclidean Parallel Postulate

- For every line l and for every point P that does not lie on l, there is exactly one line m such that P lies on m and m || l

Elliptic Parallel Postulate

- ... there is no line m such that P lies on M and m || l

Hyperbolic Parallel Postulate

- ... there are at least two lines m and n such that P lies on both m and n and both m and n are parallel to l

The parallel postulates in incidence geometry

These are not necessarily new axioms for incidence geometry

They are additional statements that may or may not be satisfied by a particular model for incidence geometry

We may choose to add one of them to incidence geometry as a further refinement (like the 5th postulate of Euclid).

- Euclidean Incidence Geometry adds the Euclidean Parallel Postulate
- Elliptical Incidence Geometry adds the Elliptical Parallel Postulate
- Hyperbolic Incidence Geometry adds the Hyperbolic Parallel Postulate

Neutral Incidence geometry

We can call the original Incidence Geometry the Neutral Incidence Geometry. We say neutral because we don't have any of the parallel postulates.

Anything true in Neutral Incidence Geometry is also true in each of the other incidence geometries (Euclidean, Elliptical, Hyperbolic).

Note that the three parallel postulates are mutually exclusive - for any Incidence Geometry, if one of the postulates is true, then the other two can't be true.

This is how we build independent geometries!

Foundations of Geometry

Neutral Geometry is Geometry without a parallel postulate. In Neutral Geometry parallel lines will be shown to exist. We just don't know if there are more than one!

- Chapter 4 - Axioms for Neutral Geometry
- Chapter 5 - Axioms for Euclidean Geometry - add to Neutral Geometry the Euclidean Parallel Postulate (Equivalent to Euclid's 5th).
- Chapter 6 - Axioms for Hyperbolic Geometry - add to Neutral Geometry the Hyperbolic Parallel Postulate which is equivalent to the negation of the Euclidean Parallel Postulate