Deductive Reasoning

Chapter 2.5, 2.6 - Theorems, Proofs, and Logic

Theorems and Proofs

- Deductive reasoning is based on strict rules that guarantee certainty
- Well, a guarantee relative to the certainty of the original assumptions
- We contrast deductive reasoning with inductive reasoning used in science
- Even an overwhelming amount of experimental evidence is not enough to reach the certainty of deductive reasoning
- We use valid rules of reasoning to construct proofs
- Writing good proofs requires practice
- A mathematical statement must be true or false, but not both. Good statements are a key to good proofs
- This chapter lays out the principles of deductive reasoning

A Valid Rule of Reasoning

Modus Ponens

Let A and B be mathematical statements. In the following argument pattern when we assert A we mean "A is true"

1. A
2. If A then B

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Therefore B

Above the line are premises of an argument pattern and below the line is the conclusion of the argument pattern. Note that it is not possible for both premises to be true and the conclusion false. We say Modus Ponens is a *valid* rule of reasoning

Proofs and Rules of Reasoning

A proof is a sequence of deductive reasoning steps starting with some statements that are taken to be true as assumptions and ending with a statement that is the desired conclusion.

Each reasoning step must valid, that is, each deductive step can only use a valid rule of reasoning like Modus Ponens. There are other valid rules of reasoning that can be used.

In all valid rules of reasoning the conclusion must be true if the premises are true. That is, it can never be the case that the premises are true and the conclusion is false in a valid rule of reasoning. This can be proven for each valid rule of reasoning.

Proofs

- The real goal is to learn to appreciate proofs
- Proofs are the hallmark of mathematics, but mathematics also involves
  - a study of patterns
  - problem solving
  - relationships among abstract entities like space and number
  - Reasoning by analogy
  - Intuition
- We look for proofs that convey both rigor and intuition without sacrificing either
- Readers of proofs must see what you're doing (intuition) and believe your arguments (rigor)

Mathematical Statements

- Mathematical statements must be precise and clear
- A mathematical statement refers to any assertion that can be classified as either true or false, but not both
- Definitions must establish the clear and precise method of determining truth of statements about mathematical entities
- Mathematical statements are based on the language of First Order Logic (FOL)
- First order logic has the following most common operations for expressing mathematical statements
    - logical connectives: and, or, not, implication
    - Existential Quantifiers: There exists, there exists a unique
    - Universal Quantifier: For all

Logical Connectives for Compound Statements

- The meaning of the logical connectives can be precisely stated using truth tables
- Truth tables for and, or, not statements
- The variables A, B represent mathematical statements (possibly themselves compound statements)

A	B	A and B	A or B	not A
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Implication Statements

- Implication is written in various ways
  - A *implies* B denoted A => B
  - If A then B (conditional)
  - B if A (hypothetical)
- In an implication *A implies B*, A is called the *Antecedent*, B is called the *Consequent*
- Theorems are expressed in the A => B form (if A then B)
- A proof of a theorem A => B is a deduction that ultimately asserts B is always true whenever A is true.
- Proofs (deductions) are constructed with repeated use of the Modus Ponens proof rule applied to axioms or previously proven theorems.
- Proofs are very closely connected with implication statements, as you can see

Truth Table for Implication Statements

A	B	A =>B	
T	T	T	
T	F	F	Counterexample (the implication is false)
F	T	T	Vacuously True
F	F	T	Vacuously True

- Implication is the basis of the modus ponens deduction rule given earlier
- A counterexample is when the implication statement is false
- The implication is vacuously true whenever the antecedent is false - regardless of the truth of the consequent (more later on this).

The negation of implication

- The truth of a statement A implies B means that whenever A is true B must be true
- Implication means that there is *no case where A is true and B is not true*
- Therefore, to negate an implication we simply need to find a case where A is true and B is not true
- We call this a *counterexample*, and only one specific case need be found
- Implication really means no counterexample allowed
- Proofs are deductions that rule out possibility of counterexample.

Common Statement Patterns in Math

Here are some common statement patterns in mathematics

- bi-implication denoted (A <=> B) means
  - *A implies B and B implies A*
- bi-implication is also written
  - *A if and only if B*
  - *A iff B*
- The contrapositive of *A implies B* is
  - *not A implies not B*
- The converse of *A implies B* is
  - *B implies A*

Vacuously True Implications

The truth of an implication is often misunderstood when the antecedent is false

- A implies B always true when A is false
- We say the implication is *Vacuously True*
- Vacuously true implications, while true, do not generally have meaning
- Here is a vacuously true statement of and by Neal (as of 2012.06.28) : "I have won the lottery every time I have played it"

Implication may be more clearly understood as an assertion that there are no counterexamples. So A implies B means that there are no cases where A is true and B is not true

Propositional Functions

Statements often have variables, for example
$P(x) = x > 0$

The statement $x > 0$ will be true or false when we supply a specific value for the variable x

Statements like $P(x)$ that depend on variables are called *propositional functions*

The variable x of a propositional function always range over some *domain* of values, typically the real numbers for our purposes. But sometimes our variables will range over geometric objects like points or lines.

Quantifiers

Whenever we have mathematical statements expressed as propositional functions, then we commonly encounter *quantifiers*

- ***(For all*** **x) P(x)** is a mathematical statement involving the universal quantifier. P(x) in this case is true for **all** x in the domain of x
- ***(There exists x)*** **P(x)** is a mathematical statement involving the existential quantifier. P(x) in this case is true for **some** x in the domain of x
- ***(There exists a unique x)*** **P(x)** is a special form of the existential quantifier that asserts P(x) is true for exactly one value of x in the domain of x

A Common Mathematical Statement

- Here is a common mathematical statement, call it S
  - For all x (P(x) implies Q(x))
- This means that S is true whenever either
  - there is no x such that P(x) is true (vacuously true) or
  - whenever P(x) is true, Q(x) is also true
- We can also say the S means there is no case where P(x) is true and Q(x) is not true
- The negation of S is, therefore, a counterexample x:
  - There exists an x such that P(x) and not Q(x)

Developing Proofs

Generally I recommend developing a proof in the following way

- Try to capture the intuition of your proof with a brief paragraph that you can use to explain your idea clearly to another classmate
- Outline your intuition with the major steps in your proof showing when you use the hypotheses and what other key theorems are needed
- Work the steps of your proof into the outline form specified in the Morgan and Breckenridge handout (Given, To Prove, Analysis, Numbered Proof Steps with Reasons)
- Work your proof into the paragraph form described by Venema in Chapter 2.6

Reductio Ad Absurdum Proofs

- Reduction Ad Absurdum (RAA) proofs are also known as proofs by contradiction, or indirect proofs
- The validity of RAA proofs is based on the following two assumptions accepted by essentially all practicing mathematicians (these are philosophical foundations)
  - The Law of the Excluded Middle, which asserts that a statement must be either true or false (not unknown)
  - The Law of Non-Contradiction, which asserts that no statement can be both true and false (what we call a logical contradiction)
- An RAA proof proves the statement *A implies B* in two steps
  1. Assume that A is false
  2. Show this assumption leads to a logical contradiction

Sample Proofs

- Euclid's Proposition 1
- Euclid's Proposition 9
- Euclid's Proposition 10
- Chapter 1, exercise 1.8
- Chapter 2.6, Thm 2.6.2 p33, an RAA proof
- A proof that the square of an even integer is even
- A proof that the square root of 2 is irrational (Apx E)
- Chapter 2.6, exercises 1
- Proof practice: Chapter 2.6, exercises 2-8