

Distance, Ruler Postulate and Plane Separation Postulate

Chapter 3.1 to 3.4 - The Axioms of
Plane Geometry

The Axioms of Plane Geometry

This Chapter 5 introduces the undefined terms and the six axioms that form the foundation of the Euclidean and Non-Euclidean geometry developed in throughout this textbook

- Venema's six axioms cover all of Euclid's Plane Geometry (ie, 2-dimensional only) that does not depend on the Euclidean Parallel postulate (Euclid's 5th postulate)
- This portion of Euclid's geometry is called *neutral geometry*
- Venema's system of axioms is based on real numbers
- Points are real numbers
- Line segments are intervals on the real number line (sets of real numbers)
- Angles are identified with their real number angle measure

Undefined Terms

- Point
- Line
- Distance
- Half-plane
- Angle Measure

Venema basically has one axiom for each undefined term plus an axiom that specifies how line segments and angles interact with each other (the SAS axiom!)

Axioms

- Existence Postulate (Points)
- Incidence Postulate (Lines and Points)
- Ruler Postulate (Distance)
- Plane Separation Postulate (Half-plane)
- Protractor Postulate (Angle Measure)
- The Side-Angle-Side Postulate (Congruence of Triangles)

Points and Lines

- The Existence Postulate gives meaning to the Euclidean plane, denoted \mathbf{P} , as a non-trivial set of points
- The Incidence Postulate gives meaning to a line as a set of points and that every pair of points determines a unique line
- Note that we have not yet indicated that points are identified with real numbers!
- We have assumed the undefined terms and axioms of set theory (so, for example, "lie-on" just means "element of")
- Notation for a line (a double-headed arrow overbar, p36)
- Definitions of *lie-on*, *incident with* and *external point*
- Definition and notation for parallel lines, $l \parallel m$
- Trichotomy of lines: If l and m are two lines, then exactly one of three conditions hold: $l = m$, $l \parallel m$, or l intersect m is precisely one point

Distance and Metric Functions

- The Ruler Postulate gives meaning to the undefined term distance
- This is where we bring in the concept of points as real numbers
- Note that we've now gone from Euclid's straight-edge to a ruler with numerical measuring marks on it
- The ruler is a function that gives a one to one correspondence between points on lines and real numbers
- The ruler function must identify every segment with a distance
- A function with the correct properties for measuring distance is called a *metric*. Def 3.2.9 p39 lists those properties
- Thm 3.2.7 p38 shows that the distance function identified in the Ruler Postulate is indeed a metric function as it must be

Metric Functions

- There are alternative ways to define a metric function
- A metric can be any function that obeys the properties of Def 3.2.9 p39
- The Ruler Postulate does not specify a particular metric function
- Typically we assume the standard Euclidean metric, which is basically the Pythagorean theorem
- The Euclidean metric is the natural metric for Plane Geometry
- The taxicab metric is an alternative to the Euclidean metric
- The taxicab metric is what we have to use in New York City when calculating distance between our hotel and restaurants

Coordinate Functions

The Ruler Postulate implicitly defines two functions

1. The metric function that measures the distance
 2. *A coordinate function* that tells us how to place the ruler
- When we place a ruler to measure a distance we can use any section of the ruler to measure
 - The easiest way to place the ruler is to put it down so that 0 is at the left end of the segment we want to measure
 - The Ruler Placement Postulate, Thm 3.2.16 p41 guarantees that it is always possible to find an appropriate coordinate function to measure a segment with a 0-aligned left side
 - Notice that the Ruler Placement Postulate is not really a postulate, but we call it that because it is the only coordinate function we really want to use

Betweenness for Points

- Def 3.2.3 - Betweenness for points is defined (p37) in terms of *distance* rather than the coordinate function directly
- Remember, according to the ruler postulate a coordinate function f assigns numbers to points
- Thm 3.2.17 - A point C is between points A and B if and only if either
 - $f(A) < f(C) < f(B)$
 - $f(A) > f(C) > f(B)$
- So the order of points is determined simply by the order of the corresponding real numbers for the points
- Using distance to define betweenness instead of numerical order avoids dependence on a particular coordinate function (see Venema p41 comments).
- Cor 3.2.20 is handy for establishing points on rays

Point Construction Postulate

- Thm 3.2.23 Point Construction Postulate - the most common application of the Ruler Postulate is to construct a point - so common we call the theorem a postulate
- When you construct a point you have to guarantee existence and uniqueness - something Euclid didn't do because he generally used a compass and straightedge for existence and uniqueness
- We rarely use the other technical details of the Ruler Postulate
- The proof is a good example of rigorous attention to details and can be used as a model

Existence and Uniqueness of Midpoints

- This is Euclid's Prop 10
- Euclid used a compass and straightedge construction
- Def 3.2.21 - We define a midpoint M as equal distance from either end
- Thm 3.2.22 - Existence and Uniqueness of Midpoints relies on the Existence and Uniqueness of the half-way point of any distance measured by a real number
- Proof of Thm 3.2.22 - Exercise 17
- The proof is an exercise in coordinate functions
- Note that you can't use the Point Construction Postulate in the proof of the Existence and Uniqueness of Midpoints because it comes after.
- But you can emulate the proof of the Point Construction Postulate

The Ruler Postulate and Continuity

- The Ruler Postulate is what guarantees that the Euclidean plane is continuous
- It does this by assuming the correspondence with real numbers and not any other type of number
- We know from the Appendix E axioms that the real numbers have the continuity property (no gaps, the Least Upper Bound axiom)
- Examples 3.2.24 and 3.2.26 very nicely illustrate the gap problem in the rational plane
- The gap problem was not explicitly understood or addressed in Euclid's work

Plane Separation

- The Plane Separation Postulate gives us the properties of two dimensions that we need to define angles and triangles
- The Plane Separation Postulate "allows us to define angle, the interior of an angle, and triangle" (Venema)
- Every line determines a half-plane and forms a boundary
- Half-planes must be convex, see Venema p47 Fig3.8
- Any line that straddles half planes must intersect at the half-plane boundary line
- The technical properties listed following the Plane Separation Postulate on p47 are important technical properties for proofs
- Prop 3.3.4 p47 restates the Plane Separation properties of a line and gives meaning to "on the same side of a line" and "on opposite sides of a line"

Rays and Angles

- Def 3.3.5 - Opposite rays, non-opposite rays
- Def 3.3.6 - Angle, vertex, sides
- Note - Opposite rays do not determine an angle, thus Venema does not allow straight angles (p48)
- Notation (Venema p47) - H_A denotes the half-plane containing point A and bounded by some line l
- Def 3.3.7 - Interior of an angle
- Exercise 1 p51 - the intersection of any two convex sets is convex, so the interior of an angle is a convex set
- Def 3.3.8, Thm 3.3.9 (p49) Betweenness for rays
- Thm 3.3.10 - Betweenness for rays and points is consistent
- Def 3.3.11 - Triangle, vertices, sides
- Theorem 3.3.12 - Pasch's Axiom

Angle Measure

The Protractor Postulate (p51) defines the measure of an angle (denoted by the Greek letter μ)

1. The measure of an angle is between 0 and 180 degrees but not including 180 degrees
2. An angle is 0 degrees if and only if its sides are equal rays.
3. Angle Construction - for every real number between 0 and 180 (not including 180) we can construct a unique angle with that measure
4. Parts of angles formed by rays between the sides add to the whole

Angles are congruent if their angle measures are equal