

The Protractor Postulate and the SAS Axiom

Chapter 3.4 - 3.7 The Axioms of
Plane Geometry

The Protractor Postulate and Angle Measure

The Protractor Postulate (p51) defines the measure of an angle (denoted by the Greek letter μ)

1. The measure of an angle is between 0 and 180 degrees but not including 180 degrees
2. An angle is 0 degrees if and only if its sides are equal rays.
3. Angle Construction - for every real number between 0 and 180 (not including 180) we can construct a unique angle with that measure
4. Parts of angles formed by rays between the sides add to the whole

Angles are congruent if their angle measures are equal

Right, Acute, Obtuse Angles

- Now we can define a right angle as the unique angle with measure 90 degrees
- Obtuse angles are ones with measure greater than 90 degrees
- Acute angles have measures less than 90

Note that Venema does not define an angle measure function, but only asserts that an angle measure exists (see Exercise 3.4.2)

Numbers are now our tools! - a ruler and a protractor - rather than a straight edge and a compass

- Segments are congruent if their lengths are equal
- Angles are congruent if their measures are equal

The Betweenness Theorem for Rays

- Recall betweenness for rays was defined by interiority of points (Def 3.3.8 p49)
- A ray is in the interior of an angle if every point in the ray is in the interior of the angle, ie, between the sides of the angle
- Thm 3.4.5 (p54) - We can now characterize betweenness numerically!
- A ray is between the sides of an angle if its angle measure (from either side) is less

Existence and Uniqueness of Angle Bisectors

- This is Euclid's Prop 9
- Euclid used a compass and straightedge construction
- Def 3.4.6 - We define an angle bisector as a ray between the sides that makes an equal angle measure with both sides
- Thm 3.4.7 - Existence and Uniqueness of Angle Bisectors relies on the Existence and Uniqueness of the half-way point of any angle measured by a real number
- Proof of Thm 3.4.7 - Exercise 3.4.1
- The proof is an exercise in using the Protractor Postulate

The Crossbar Theorem

- Thm 3.5.2 The Crossbar Theorem - Any ray in the interior of an angle of a triangle intersects with the side opposite the angle. The ray must cross the bar
- Theorem 3.3.10 is related to the converse of the Crossbar Theorem - a point D is interior to the crossbar iff a ray from the vertex through the point D is interior to the angle
- Thm 3.5.3 combines the two into one theorem and summarizes the consistent meaning of both points and rays in the interior of an angle
- Thm 3.5.3 - A point D is interior to an angle if and only if a ray through D from the vertex intersects the interior of the crossbar
- *The Crossbar Theorem can be taken as an axiom if you have no patience for the proof*

The Linear Pair Theorem

- Def 3.5.4 - Opposite rays form a linear pair (Venema p58)
- Thm 3.5.5 Linear Pair Theorem - If two angles form a linear pair then their angle measures sum to 180 degrees
- Def 3.5.6 Two angles are *supplements* if their angle measures sum to 180 degrees
- Thm 3.5.5 Linear Pair Theorem - If two angles form a linear pair then they are supplements
- This supplementary angle theorem is effectively a completion of the protractor postulate
- It tells us how to deal with 180 degrees
- The *Linear Pair Theorem can be taken as an axiom* if you have no patience for the proof

Perpendicular Bisectors

- Def 3.5.8 - Two lines are perpendicular if they form a right angle with each other
- Def 3.5.10 - A perpendicular bisector of a segment is a line perpendicular to the line determined by the segment that also intersects with the midpoint of the segment
- Thm 3.5.9 - Existence and Uniqueness of raising a perpendicular from point on a line (Proof - Exercise 3.5.2)
- Thm 3.5.11 - Existence and Uniqueness of Perpendicular Bisectors (Proof - Exercise 3.5.3)
- Protractor postulate guarantees the existence and uniqueness of right angles, so perpendiculars must be unique

Vertical Angles Theorem

- Thm 3.5.13 - Vertical Angles are congruent
- Proof of Thm 3.5.13 - Exercise 3.5.5
- Possibly use the Linear Pair theorem and subtract out the angle common to both linear pairs

The SAS Postulate

- Triangles are congruent if all their parts are congruent (Venema p63)
- SAS - Side Angle Side triangle congruence condition
- Example 3.6.2 - SAS fails in the taxicab geometry
- The SAS Axiom (Venema p64)
- A sweet proof of the Isosceles Triangle Theorem (p65 Venema)

The Parallel Postulates

- Euclidean Parallel Postulate
 - For every line l and for every point P that does not lie on l , there is **exactly one** line m such that P lies on m and $m \parallel l$
- Elliptic Parallel Postulate
 - For every line l and for every point P that does not lie on l , there is **no** line m such that P lies on m and $m \parallel l$
- Hyperbolic Parallel Postulate
 - For every line l and for every point P that does not lie on l , there are **at least two** lines m and n such that P lies on both m and n and both m and n are parallel to l

Models for our Geometries

- Ruler Postulate rules out finite models from Chapter 2
- Sphere \mathbf{S}^2 does not satisfy Incidence Postulate or Ruler Postulate (why not the Ruler Postulate?)
- Rational Plane does not satisfy the Ruler Postulate
- Cartesian Plane with Taxicab Metric fails to satisfy SAS

- Cartesian Plane with Euclidean Metric OK so far
- Klein Disk OK so far

Diagrams illustrate relationships and are one possible interpretation of a theorem. There may be others

Theorems and proofs do not, however, depend on the diagrams