Distinguishing Euclidean and Hyperbolic Properties

Chapter 4.5 - 4.9 Neutral Geometry

Angle Sum for Triangles

- Angle sums turn out to be a distinguishing characteristic of Euclidean vs non-Euclidean geometries.
- Thm 4.5.2 (Saccheri-Legendre Theorem) - In Neutral Geometry we can prove that the angle sum for triangles is less or equal to 180.
- Note that we haven't ruled out triangle sums less than 180 - we haven't proven that they can't be less than 180
- Neutral Geometry will bifurcate into
  - Euclidean Geometry with angle sums equal to 180 on the one hand and
  - Hyperbolic Geometry with angle sums strictly less than 180
- Triangle sums on the Sphere S^2 are strictly greater than 180

Converse to Euclid's Fifth Postulate

- Euclid's Fifth Postulate is not provable in Neutral Geometry
- Cor 4.5.7 (Converse to Euclid's Fifth Postulate) is provable in Neutral Geometry
- The proof is a direct application of the previous Corollary 4.5.6.

Quadrilaterals

- We're really only interested in *convex* quadrilaterals as shown on the left side of Fig 4.25 p88
- The order of edges denoting a quadrilateral is important
- Thm 4.6.4 (p89) The angle sum of a (convex) quadrilateral is less or equal to 360 in Neutral Geometry.
- Note that in Neutral Geometry we haven't ruled out quadrilaterals with angle measure less than 360
- Angle sums for quadrilaterals, like triangles, are a distinguishing characteristic of Euclidean vs Hyperbolic geometry
- Quadrilateral angle sums strictly less than 360 signify and characterize Hyperbolic Geometry
- Quadrilateral angle sums equal to 360 signify and characterize Euclidean Geometry

Euclidean Parallel Postulate Equivalents

- The Euclidean Parallel Postulate (EPP) cannot be proven in Neutral Geometry because it is independent of the other postulates
- We can prove in Neutral Geometry that several propositions are equivalent to the Euclidean Parallel Postulate
- Propositions (Theorems) are equivalent if each can be assumed and lead to a successful proof of the other
- Chapter 5 p108 summarizes all the theorems that are equivalent to the EPP.
- Any one of the EPP equivalents can be assumed and the others follow as a consequence
- Chapter 4 has the proofs of equivalence

Defect in Triangles and Quadrilaterals

- Angle sums for quadrilaterals and triangles are a distinguishing characteristic of Euclidean vs Hyperbolic geometry
- Recall, we're only interested in convex quadrilaterals (see fig 4.25, 4.26)
- A *defect* (p98) for a triangle (or rectangle) is the difference between the angle measure and 180 (360 for rectangles)
- A rectangle is a quadrilateral with only right angles (p98)
- Thm 4.8.4 (p98) A geometry is Euclidean if and only if
  - No triangle has defect
  - No rectangle has defect
  - There exists a rectangle!
- These are also equivalent to the Euclidean Parallel Postulate

Rectangles and Euclidean Geometry

- So a geometry is Euclidean if and only if there exists a rectangle!
- Some insightful historical attempts to prove the fifth postulate started by assuming the fifth postulate was false and then studying rectangles whose angle sums are less than 360
- Following the historical discovery of non-Euclidean geometry, here are two important rectangle-like quadrilaterals that facilitate our study of Hyperbolic Geometry in Chapter 6
  - Saccheri quadrilateral in Fig 4.42 (p102)
  - Lambert quadrilateral in Fig 4.43 (p102)
- Read and bookmark Thm 4.8.10 (p103) for later reference
- Read and bookmark Thm 4.8.11 (p103) for later reference

The Universal Hyperbolic Theorem

- The Hyperbolic Parallel Postulate (HPP) asserts for every line l and every external point P there are multiple parallel lines through P (at least two)
- Is it possible that some lines and external points have unique parallels and others have multiple parallels?
- In other words, is the multiple parallel property universal for lines and external points?
- Thm 4.9.1 (The Universal Hyperbolic Theorem) - If any line and external point P has the multiple parallel property then all lines and external points have the multiple parallel property
- The HPP is a universal property for a geometry
- Cor 4.9.3 (p105) - In any model for Neutral Geometry either the EPP or the HPP will hold, but not both.