Euclidean Geometry

Chapter 5.1 - 5.4 Euclidean Geometry

Theorems Unique to Euclidean Geometry

- We first assume the Euclidean Parallel Postulate, EPP p 21
- With the EPP we get for free all theorems equivalent to the EPP proved in Chapter 4 and listed on p108 of Chapter 5
  - Alternate interior angles between parallel lines are congruent (Converse to the Alt Interior Angles Thm)
  - Euclid's Postulate 5
  - Triangles sums are exactly 180
  - Quadrilateral sums are exactly 360
  - Similar triangles exist (Wallis' postulate p96)
  - Transitivity of parallelism (Thm 4.7.3)
  - Existence of Rectangles (Clairaut's Axiom 4.8.7)
  - Plus some facts about perpendicularity (Theorem 4.7.3)

Key Results Chapter 5.1 - 5.4

- Section 5.2 - the Parallel Projection Theorem establishes the key technical result needed for the Similar Triangles Theorem
- Section 5.3 - the Similar Triangles Theorem
  - Similar triangles are unique to Euclidean Geometry
  - Similarity does not exist in Hyperbolic Geometry
- Section 5.4 - The Pythagorean Theorem
  - The proof is based on similar triangles
  - "It is thought that early Greek proofs were also based on similar triangles" - Venema p107 top
  - The earliest Greek proofs were correct only for the case of commensurate lengths
  - Commensurate lengths are those that can be measured by a common unit, that is, their ratios are rational numbers

Parallel Projection Theorem

- This proof is worth studying closely and understanding
- The name of the theorem alludes to the diagram on p110 where the parallel lines preserve ratios as one transversal is projected onto the other
- The proof proceeds in three major cases
  - The parallel lines preserve equality (Figure 5.4 p111)
  - The ratios on the transversals are rational (Figure 5.5 p112)
  - Finally, the ratios on the transversals can be irrational (see figure 5.6 p112)
- To prove the irrational case we appeal to the Comparison Theorem E.3.3

Parallel Projection - The Rational Case

- The proof of the equal projection in Lemma 5.2.2 is a straightforward application of triangle congruences - study it
- The proof of the rational case of Parallel Projection is based on breaking up the ratio on the first transversal into equal sized parallels and then applying the above equal projection case over and over.
- Breaking the ratio on the first transversal into equal sized parallels is possible because the two lengths are commensurable
- Two lengths are commensurable if they can be measured by a common unit
- The ratio of commensurable lengths is therefore always a rational number

Parallel Projection - The Irrational Case

- The proof of the irrational case (bottom paragraph of p111) is a an excellent example of how to apply of the Comparison Theorem E.3.3 using the rational proof case
  - Every rational approximation ***x*** less than the first ratio is also a rational approximation less than the second ratio **y**
  - Similarly every rational approximation ***y*** less than the second ratio is a also a rational approximation less than the first ratio **x**
- The technique of rational approximation embedded in this theorem and proof was discovered by Edoxus just before Euclid's time and comprises the 5th book of Euclid

The Similar Triangles Theorem

- The proof of this theorem is fairly straightforward using the parallel projection theorem
- Note the need to use Pasch's Theorem to assert the existence of the point C'.
- The desired similarity property for the similar triangles theorem is obtained in the last step by "cross multiplying" ratios. We justify this because we have algebra!
- The Birkhoff Axioms in Appendix B2 p400 take the Side-Angle-Side Similarity Criterion of theorem 5.3.3 as an axiom to substitute for both the Side-Angle-Side Congruence Axiom and the EPP. Note that Birkhoff has only 4 axioms! (Line Measure, Angle Measure, Incidence, Similarity).

The Pythagorean Theorem (Sec 5.4)

- The proof in Venema uses the Similar Triangles Theorem
- The proof introduces some short-cut notation in labeling segments directly with variables as we commonly do in algebra. We know this is OK because we identify lengths with real numbers by the Ruler Postulate
- The proof is simple, but uses a clever bit of algebra
- First the Similar Triangles Theorem is applied twice
- Then the resulting two equal ratios are cross multiplied to get two equalities, each with one of the desired terms a^2 and b^2
- Finally, the two equations are jointly added to get the desired result.
- This concludes the foundations of Euclid's Book I which ends with the Pythagorean Theorem