Hyperbolic Geometry

Chaper 6.1 - 6.6 Hyperbolic Geometry

Theorems Unique to Hyperbolic Geometry

- Now we assume Hyperbolic Parallel Postulate (HPP - p21)
- With the HPP we get for free all the ***negations*** of the theorems equivalent to the EPP proved in Chapter 4 and listed on p108 of Chapter 5. Here's a few samples
  - Triangle sums are strictly less than 180 (6.1.1)
  - Quadrilateral sums are strictly less than 360 (6.1.3)
  - There does not exist a rectangle (6.1.6)
  - Similar Triangles do not exist unless they are congruent (AAA Congruence 6.1.11)
- There are two kinds of parallel lines in Hyperbolic Geometry
  - Asymptotic parallels (6.6.2)
  - Divergent parallels with Common Perpendicular (6.6.2)
- All lines we draw are straight, even though they won't look straight when we draw them on a Euclidean surface.

All Neutral Geometry Theorems Hold in Hyperbolic Geometry

- Our drawings of Hyperbolic Geometry on Euclidean surface might provoke our skepticism about the truth of relations in the diagrams - they just don't look right
- But we are studying Hyperbolic Geometry axiomatically, so all of the theorems of Neutral Geometry still hold, even if the pictures don't look right
- For example, there is exactly one one line through two points (incidence axiom)
- Diagrams are (only) for the purpose of aiding our understanding of the names and relationships of the objects discussed in the theorems and proofs
- Truth is established by the proofs, not the pictures

Quadrilaterals and Triangles in Hyperbolic Geometry

- Properties of Saccheri quadrilaterals - read these
  - Summit angles are acute (Cor 6.1.4)
  - Length of Altitude is less than length of side (Cor 6.1.9)
  - Length of Summit is greater than length of base (Cor 6.1.10)
  - Can't scale Sacherri quadrilaterals just like you can't scale triangles (Thm 6.1.12)
- Properties of Lambert Quadrilaterals- read these
  - The fourth angle is acute (Cor 6.1.5)
  - The length of a side between two right angles is strictly less than the length of the opposite side (Thm 6.1.7)

AAA Congruence in Hyperbolic Geometry

- If two triangles are similar, then they are congruent (Thm 6.1.11)
- The proof is an RAA proof. Good practice reading this.
- Notice the phrase "without loss of generality". This means that the diagrams can simply be re-labeled to make sure the assumed relationships hold

Common Perpendiculars in Hyperbolic Geometry

- Parallel lines in Hyperbolic Geometry do not stay the same distance apart
- It's even more dramatic - Thm 6.2.1 shows that there can be no three points on any one line equidistant from any other line. (But there might be two).
- A common perpendicular is illustrated in Figure 6.10 p139. See Def 6.2.2.
- Common perpendiculars are unique (Thm 6.2.4)
- If there are two points on a line equidistant from another line, then the two lines admit a common perpendicular (Thm 6.2.3)
- These are nice theorems to prove (Exercises 6.2.1, 6.2.2)
- Converse to the Alternate Interior Angles Theorem only holds in a special case (Thm 6.2.5, Figure 6.11 p140)

Limiting Rays and Asymptotically Parallel Lines

- Sections 6.3 and 6.4 contain technical material needed for further study of Hyperbolic Geometry
- These sections set up the definitions and basic properties of two distinct kinds of parallels
  - Asymptotic Parallels that do not admit a common perpendicular
  - Divergent Parallels that admit a common perpendicular
- The summary along with Figure 6.33 (pp 154-55) is very helpful intuition to guide your reading - read this first
- The *Critical Number* discussed on p141 is introduced to properly define the *Angle of Parallelism* based on the Least Upper Bound Postulate (Axiom E.2 p368)
- The Angle of Parallelism is given in Def 6.3.4 p141 and leads to *Limiting Parallel Rays* in Section 6.4 p144

The Angle of Parallelism

- We're given a line l, an external point P, and a perpendicular dropped from P to l.
- We know that there is a plenitude of lines through P parallel to l.
- We also know that there is a plenitude of lines through P that intersect with l (see Figure 6.13 p142).
- We're looking for the line m through P that is *closest* to l than all others, but does not intersect l.
- We identify the parallel lines through P parallel with l by their angle with the perpendicular
- The critical number r_0 is the angle measure of the *angle of parallelism* of the line through P that is parallel to l but with no parallel line "closer" to l.

The Critical Number of the Angle of Parallelism

- Assume l is a line and P is an external point to l
- Assume m is a line through P. The line m may or may not intersect l
- The critical number is the Least Upper Bound of the set of angle measures for lines m that intersect l
- This means the line m with the critical number r_0 angle measure does not intersect l, but every line with angle measure less than r_0 does intersect l
- The critical number cannot be 90 or we would be in Euclidean Geometry (Thm 6.3.8 p144)
- The critical number depends only on the distance from P to l (Thm 6.3.5 p142)!
- As P recedes from l, the critical number decreases (Thm 6.3.7 p143)

Limiting Parallel Rays

- The critical number determines the angle of parallelism and gives meaning to Limiting Parallel Rays
- Limiting Parallel Rays are a new kind of parallelism that arises in hyperbolic geometry (Def 6.4.1 p144)
- Two limiting parallel rays determine two asymptotic parallel lines (Thm 6.4.2)
- Limiting Parallelism of rays is symmetric (Thm 6.4.3)
- Limiting Parallelism of rays is transitive (Thm 6.4.7)
- For a line l and a point P not on l there is a unique ray through P limiting parallel to l (Thm 6.4.5)
- Limiting parallelism depends on limiting behavior of the rays, not the endpoint where the ray begins (Thm 6.4.4)

Asymptotic Triangles

Section 6.5 describes a new kind of triangle-like object in hyperbolic geometry that does not exist in Euclidean Geometry

- An Asymptotic triangle consists of two limiting parallel rays and a segment joining the endpoints of the rays (Def 6.5.1)
- Asymptotic triangles only have three parts, not 6 like triangles
- Congruence of asymptotic triangles requires correspondence of the three parts
- Congruence for asymptotic triangles can be established with only two congruent parts - Side Angle, or Angle Angle
- Asymptotic triangles have their own Exterior Angle Theorem (6.5.2) and Angle Sum Theorem (6.5.3).

Classification of Parallels

- Thm 6.6.2 (p152) - there are two distinct kinds of parallel lines
  - Those that do not admit a common perpendicular and are called asymptotic parallel lines
  - Those that do admit a common perpendicular and I call divergent parallel lines
- There are *no other* kinds of parallel lines than asymptotic parallel lines or divergent parallel lines.
- For a line l and a point P not on l there are two asymptotically parallel lines through P (one in each direction) and an infinite number of divergent parallel lines through P between the two asymptotically parallel lines - see Figure 6.3.3 p155