Distinguishing Euclidean and Hyperbolic Properties

Chapter 4.5 - 4.9 Neutral Geometry
Angle Sum for Triangles

- Angle sums turn out to be a distinguishing characteristic of Euclidean vs non-Euclidean geometries.
- Thm 4.5.2 (Saccheri-Legendre Theorem) - In Neutral Geometry we can prove that the angle sum for triangles is less or equal to 180.
- Note that we haven't ruled out triangle sums less than 180 - we haven't proven that they can't be less than 180.
- Neutral Geometry will bifurcate into
  - Euclidean Geometry with angle sums equal to 180 on the one hand and
  - Hyperbolic Geometry with angle sums strictly less than 180.
- Triangle sums on the Sphere $S^2$ are strictly greater than 180.
Converse to Euclid's Fifth Postulate

- Euclid's Fifth Postulate is not provable in Neutral Geometry
- Cor 4.5.7 (Converse to Euclid's Fifth Postulate) is provable in Neutral Geometry
- The proof is a direct application of the previous Corollary 4.5.6.
Quadrilaterals

- We're really only interested in convex quadrilaterals as shown on the left side of Fig 4.25 p88
- The order of edges denoting a quadrilateral is important
- Thm 4.6.4 (p89) The angle sum of a (convex) quadrilateral is less or equal to 360 in Neutral Geometry.
- Note that in Neutral Geometry we haven't ruled out quadrilaterals with angle measure less than 360
- Angle sums for quadrilaterals, like triangles, are a distinguishing characteristic of Euclidean vs Hyperbolic geometry
- Quadrilateral angle sums strictly less than 360 signify and characterize Hyperbolic Geometry
- Quadrilateral angle sums equal to 360 signify and characterize Euclidean Geometry
The Euclidean Parallel Postulate (EPP) cannot be proven in Neutral Geometry because it is independent of the other postulates.

We can prove in Neutral Geometry that several propositions are equivalent to the Euclidean Parallel Postulate.

Propositions (Theorems) are equivalent if each can be assumed and lead to a successful proof of the other.

Chapter 5 p108 summarizes all the theorems that are equivalent to the EPP.

Any one of the EPP equivalents can be assumed and the others follow as a consequence.

Chapter 4 has the proofs of equivalence.
Defect in Triangles and Quadrilaterals

- Angle sums for quadrilaterals and triangles are a distinguishing characteristic of Euclidean vs Hyperbolic geometry.
- Recall, we're only interested in convex quadrilaterals (see fig 4.25, 4.26).
- A defect (p98) for a triangle (or rectangle) is the difference between the angle measure and 180 (360 for rectangles).
- A rectangle is a quadrilateral with only right angles (p98).
- Thm 4.8.4 (p98) A geometry is Euclidean if and only if:
  - No triangle has defect
  - No rectangle has defect
  - There exists a rectangle!
- These are also equivalent to the Euclidean Parallel Postulate.
Rectangles and Euclidean Geometry

● So a geometry is Euclidean if and only if there exists a rectangle!
● Some insightful historical attempts to prove the fifth postulate started by assuming the fifth postulate was false and then studying rectangles whose angle sums are less than 360
● Following the historical discovery of non-Euclidean geometry, here are two important rectangle-like quadrilaterals that facilitate our study of Hyperbolic Geometry in Chapter 6
  ○ Saccheri quadrilateral in Fig 4.42 (p102)
  ○ Lambert quadrilateral in Fig 4.43 (p102)
● Read and bookmark Thm 4.8.10 (p103) for later reference
● Read and bookmark Thm 4.8.11 (p103) for later reference
The Universal Hyperbolic Theorem

- The Hyperbolic Parallel Postulate (HPP) asserts for every line \( l \) and every external point \( P \) there are multiple parallel lines through \( P \) (at least two).
- Is it possible that some lines and external points have unique parallels and others have multiple parallels?
- In other words, is the multiple parallel property universal for lines and external points?
- Thm 4.9.1 (The Universal Hyperbolic Theorem) - If any line and external point \( P \) has the multiple parallel property then all lines and external points have the multiple parallel property.
- The HPP is a universal property for a geometry.
- Cor 4.9.3 (p105) - In any model for Neutral Geometry either the EPP or the HPP will hold, but not both.